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On semigroups of normal matrices

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Abstract

Semigroups of normal complex square matrices having a finite spectrum are studied. It is shown that every member A of such a semigroup satisfies the condition $A^* = A^n$, where n does not depend on the matrix A. Further, it is shown that such a semigroup is completely reducible, and that each irreducible contraction consists of a group of unitary matrices and, eventually, the zero matrix.

Keywords: Normal matrices; Spectrum; Semigroups; Reducibility

1. Introduction

One of the equivalent conditions for a complex square matrix A to be normal is that it satisfies the equation $A^* = p(A)$ for some polynomial p, where A^* denotes the adjoint of A (see [2, Condition 17]). Let $\mathscr{S}$ always denote a multiplicative semigroup of normal complex matrices. According to the above condition, every matrix A of a semigroup $\mathscr{S}$ satisfies the equation $A^* = p_A(A)$. Let the polynomial p be fixed. If every member A of a semigroup $\mathscr{S}$ satisfies $A^* = p(A)$, then the spectrum of such a semigroup

$$\sigma(\mathscr{S}) = \bigcup_{A\in\mathscr{S}} \sigma(A)$$

with $\sigma(A)$ being the spectrum of the matrix A is finite. The only exception is a semigroup of hermitian matrices, where $p(\cdot) = \mathrm{id}(\cdot)$ (that is, the identity transformation). On the other hand, the finiteness of the spectrum of the semigroup $\mathscr{S}$ implies

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the existence of a fixed positive integer n such that every matrix $A \in \mathscr{S}$ satisfies $A^* = A^n$. This will be shown in Section 2.

In Section 3 we will be concerned with the reducibility of semigroups of normal matrices. Viewing matrices as operators on a finite-dimensional space $\mathscr{V}$, the semigroup of matrices $\mathscr{S}$ is said to be *irreducible* if no nontrivial subspace of $\mathscr{V}$ invariant under all members of $\mathscr{S}$ exist. The semigroup $\mathscr{S}$ is said to be *completely reducible* if it is a direct sum of irreducible subsemigroups. We will show that every semigroup of normal complex matrices $\mathscr{S}$ is completely reducible, and that each irreducible contraction (also called the irreducible component) of $\mathscr{S}$ consists of multiples of unitary matrices. In the case of the semigroup with finite spectrum each nonzero irreducible contraction consists of a group of unitary matrices and, possibly, the matrix 0.

2. Semigroups with finite spectrum

A matrix A satisfying the equation $A^* = A^n$ for some $n \geqslant 1$ is said to be *power-hermitian* (compare [3]). In this section we will be concerned with the equivalence between the semigroups having finite spectrum and the semigroups of power-hermitian matrices (of the same power). Because of slight differences in conditions regarding hermitian matrices, this result will be presented in two separate propositions.

Proposition 2.1. *Let p be a fixed polynomial in one variable, not equal to the identity transformation, i.e. $p(x) \neq x$. If all members A of the semigroup $\mathscr{S}$ satisfy the equation $A^* = p(A)$, then the spectrum of $\mathscr{S}$ is finite.*

Proof. Since $A^* = p(A)$ for all $A \in \mathscr{S}$, $\mathscr{S}$ being a semigroup of normal matrices, the spectrum of $\mathscr{S}$ satisfies $\sigma(\mathscr{S}) \subseteq \{\lambda \in \mathbf{C};\ \overline{\lambda^k} = p(\lambda^k),\ k \in \mathbf{N}\}$. Consider the special case for $k = 2$. Every $\lambda \in \sigma(\mathscr{S})$ must satisfy $\overline{\lambda^2} = p(\lambda^2)$, or $(p(\lambda))^2 = p(\lambda^2)$. This is a polynomial equation for λ. If the equation is not the identity, it has a finite number of solutions, thus $\sigma(\mathscr{S})$ is finite. By induction on the degree of the polynomial p it can be shown that the above equation represents the identity for two different polynomials only. One of them is the zero polynomial $p(\lambda) \equiv 0$, in which case $\mathscr{S} = \{0\}$. The other one is the monomial $p(\lambda) = \lambda^n$, where $n \geqslant 0$. Depending on the degree n, three different types of the resulting semigroup $\mathscr{S}$ can appear.

1. If $n = 0$, then $p(\lambda) = 1$ and $\mathscr{S} = \{I\}$, where I denotes the identity matrix.
2. If $n = 1$, then $p(\lambda) = \lambda$ and the resulting semigroup consists of hermitian matrices, which can have the infinite spectrum. This case was excluded by assumption.
3. If $n \geqslant 2$, then the spectrum of the resulting semigroup $\sigma(\mathscr{S}) \subseteq \{0, \exp(\mathrm{i}k2\pi/(n+1)), k = 0, 1, \dots, n\}$, which is finite. $\square$

In the opposite direction, the finiteness of the spectrum of the semigroup $\mathscr{S}$ not only implies the existence of the fixed polynomial p, but also its form. Also, it does not exclude hermitian matrices.

Proposition 2.2. *If the semigroup $\mathscr{S}$ of normal complex matrices has a finite spectrum, then $\mathscr{S}$ consists of power-hermitian matrices of the form $A^* = A^n$ for some fixed $n \geqslant 1$.*

Proof. Since the zero matrix satisfies $0^* = 0^n$ for all $n \geqslant 1$, we will assume from now on that $\mathscr{S} \neq \{0\}$. Every $\lambda \in \sigma(\mathscr{S})$ implies $\lambda^k \in \sigma(\mathscr{S})$ $(k \geqslant 2)$, and since $\sigma(\mathscr{S})$ is finite, we have $\sigma(\mathscr{S}) \subset \{\lambda \in \mathbf{C};\ |\lambda| = 0 \text{ or } |\lambda| = 1\}$. Among subsequent powers of $0 \neq \lambda \in \sigma(\mathscr{S})$ there must be $\lambda^i = \lambda^j$ for some $i < j$, and consequently $\lambda^{j-i} = 1$, thus $1 \in \sigma(\mathscr{S})$. The above difference $j - i < p$, if $\sigma(\mathscr{S})$ contains p elements. Thus, every $0 \neq \lambda \in \sigma(\mathscr{S})$ satisfies $\lambda^{m_\lambda} = 1$, where $m_\lambda < p$. Let m be the least common multiple of $\{m_\lambda\}_{0 \neq \lambda \in \sigma(\mathscr{S})}$. Then $\lambda^m = 1$ and $\lambda^{m-1} = \lambda^{-1} = \overline{\lambda}$ for all $0 \neq \lambda \in \sigma(\mathscr{S})$. We thus found the exponent $n = m - 1$, independent of λ and such that $\overline{\lambda} = \lambda^n$ for all $\lambda \in \sigma(\mathscr{S})$. From this and because normal matrices are diagonalizable the desired relation $A^* = A^n$ follows. □

Remember that a *group inverse* of an arbitrary square matrix A is defined to be the (unique) matrix $A^\#$ which satisfies the equations (see [1, p. 162])

- $AA^\#A = A$,
- $A^\#AA^\# = A^\#$,
- $AA^\# = A^\#A$.

The group inverse of the matrix A proves to be the inverse of the matrix A, contained in any group containing A. Further, in normal matrix case the group inverse and the Moore–Penrose inverse of the matrix A are the same.

Corollary 2.3. *Each element of the semigroup $\mathscr{S}$ of normal complex matrices with finite spectrum has a group inverse contained in $\mathscr{S}$.*

Proof. The assertion follows from the diagonalizability of every $A \in \mathscr{S}$ and the fact that every nonzero eigenvalue of A satisfies $\overline{\lambda} = \lambda^n = \lambda^{-1}$. □

3. Reducibility of semigroups of normal matrices

As known, a semigroup of hermitian matrices is commutative and thus simultaneously diagonalizable. Our aim is to show that a semigroup of power-hermitian matrices of the same power $n \geqslant 2$ is simultaneously block-diagonalizable, the diagonal blocks being either unitary or zero.

Let us start with a general result on reducibility of semigroups of normal matrices. This result can probably be derived from known results of representation theory. However, we include a self-contained proof, for the sake of completeness.

Theorem 3.1. *Any semigroup $\mathscr{S}$ of normal complex $m \times m$ matrices is completely reducible. Each irreducible component consists of scalar multiples of unitary matrices.*

Proof. If all members of $\mathscr{S}$ have all their eigenvalues of the same absolute value, then $\mathscr{S}$ is the semigroup of multiples of unitary matrices, which might not be reducible. If it is reducible, it is completely reducible by normality.

In the opposite case take the matrix $M \in \mathscr{S}$ having at least two absolutely different eigenvalues. Let all matrices be given in the basis of the underlying space, in which M is of the form

$$M = r\begin{pmatrix} U & 0 \\ 0 & \Lambda \end{pmatrix},$$

where r is the spectral radius of M, U is the unitary (diagonal) square matrix of dimension $k < m$, and Λ is a diagonal matrix with diagonal entries λ_i, $|\lambda_i| < 1$ for $i = 1, 2, \ldots, m-k$. Let

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathscr{S}$$

be an arbitrary conformally partitioned matrix. Since the matrix X is normal, it follows that $AA^* + BB^* = A^*A + C^*C$. For the same reason, since MX is normal, we have $UAA^*U^* + UBB^*U^* = A^*A + C^*\Lambda^*\Lambda C$. Combining the two gives $U(A^*A + C^*C)U^* = A^*A + C^*\Lambda^*\Lambda C$. After equating the traces we get $\operatorname{tr}(C^*C) = \operatorname{tr}((\Lambda C)^*(\Lambda C))$, or $\sum_{i=1}^{m-k}\sum_{j=1}^{k}|c_{ij}|^2 = \sum_{i=1}^{m-k}\sum_{j=1}^{k}|\lambda_i c_{ij}|^2$, c_{ij} being the entries of the submatrix C. Further we get $\sum_{i=1}^{m-k}(1-|\lambda_i|^2)\sum_{j=1}^{k}|c_{ij}|^2 = 0$, and since $1 - |\lambda_i|^2 > 0$ for all $i = 1, 2, \ldots, m-k$, all entries c_{ij} of the submatrix C are equal to 0. Thus,

$$X = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

and the normality of X implies that $B = 0$ as well. So, every $X \in \mathscr{S}$ is in block-diagonal form

$$X = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

After repeating the above step on the resulting contractions, if necessary, the semigroup $\mathscr{S}$ comes out to be a direct sum of semigroups, each consisting of multiples of unitary matrices. $\square$

Corollary 3.2. *Any semigroup of normal complex matrices $\mathscr{S}$ with finite spectrum is completely reducible. Each nonzero irreducible component consists of a group of unitary matrices and, eventually, the matrix zero.*

Proof. Since this corollary is a special case of the Theorem 3.1, all assertions of the previous theorem are valid. In addition, the finiteness of the spectrum of the semigroup $\mathscr{S}$ implies that each irreducible contraction contains unitary matrices only, and perhaps the matrix 0. Let $\mathscr{S}'$ be an arbitrary nonzero irreducible contraction of the semigroup $\mathscr{S}$. Take a unitary $U \in \mathscr{S}'$. As in the proof of Proposition 2.2 we can see that $U^m = I$ and $U^{m-1} = U^* = U^{-1}$ for some positive integer m. Consequently, $\mathscr{S}'$ contains the identity matrix and the inverse of each of its unitary member and is thus, after removing the zero matrix if present, a group. □

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